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LOCALIZATION OF THE ATTRACTORS OF THE NON-AUTONOMOUS LIÉNARD EQUATION BY THE METHOD OF DISCONTINUOUS COMPARISON SYSTEMS†

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The construction of simple discontinuous comparison systems with non-linear elements of the "dry friction" type is proposed. The sets of closed trajectories of such systems, which are contact-free with respect to the vector field of the initial system, enable one to obtain estimates of the dissipation domain simply. A similar approach is also used to construct annular domains. The absence of the property of convergence in the case of a Liénard system with a periodic additive term follows from the existence of these domains.

There are many results on the dissipative nature of the non-autonomous Liénard equation [1–3]. Many estimates of the dissipation domain are based on considerations of the energy integral which, in a certain part of the phase space, possesses the properties of a Lyapunov function. In the other parts of phase space, it is found to be necessary to carry out additional special constructions and estimates along the trajectories being considered. All of this makes it difficult to obtain effective estimates of the global attractors of the Liénard equation. The use of the trajectories of discontinuous comparison systems rather than the energy integral enable one to avoid these difficulties and to formulate theorems on the localization of the attractors of the non-autonomous Liénard equation.

Consider the system

$$\frac{dy}{dt} = -\mu(F(y) - E(t)) - x, \quad \frac{dx}{dt} = y \tag{1}$$

where $F(y)$, $E(t)$ are functions which satisfy the Lipschitz condition and μ is a positive number.

We shall subsequently assume that the inequalities

$$\alpha\mu > 2, \quad \frac{F(y) - E(t)}{y} > \frac{\alpha y - k\,\mathrm{sign}\, y}{y}, \quad \forall t \in R^1, \quad \forall y \neq 0 \tag{2}$$

are satisfied for certain positive numbers α and k.

Assumption (2) is quite natural and traditional for Liénard systems [1–3].

We now consider the linear systems

$$\frac{dy}{dt} = -\mu\alpha y - x + \mu k, \quad \frac{dx}{dt} = y \tag{3}$$

$$\frac{dy}{dt} = -\mu\alpha y - x - \mu k, \quad \frac{dx}{dt} = y \tag{4}$$

and, also, the positive half-trajectory of system (3) with the initial data $y(0) = 0, x(0) = -\mu k$ and the positive half-trajectory of (4) with the initial data $y(0) = 0, x(0) = \mu k$ (Fig. 1).

The corresponding trajectory solutions for $G_1(x, \mu k)$ and $G_2(x, \mu k)$ are given by

$$G\frac{dG}{dx} = -\mu\alpha G - x + \mu k, \quad G\frac{dG}{dx} = -\mu\alpha G - x - \mu k \tag{5}$$

The set

$$\Omega(\alpha, k) = \{x \in [-\mu k, \mu k], \ G_2(x, \mu k) \leqslant y \leqslant G_1(x, \mu k)\}$$

is now introduced into the treatment.

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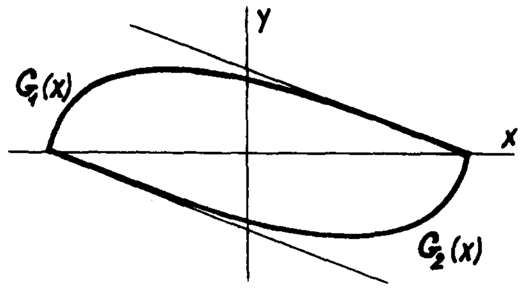

Fig. 1.

We recall that the invariant attracting set is called the attractor of system (1). If the attraction domain of an attractor when $t \to \infty$ is the whole of the phase space R^2, then such an attractor is called a global attractor.

Theorem 1. The global attractor of system (1) is contained in the set $\Omega(\alpha, k)$.

Proof. Let inequality (2) be satisfied for $k = k_0$. It is obvious that it is also satisfied for all $k \geqslant k_0$. But, then, for any point (x_0, y_0) of the set $R^2 \backslash \Omega(\alpha, k_0)$, a number $k \geqslant k_0$ exists such that (x_0, y_0) belongs to the boundary $\Omega(\alpha, k)$. Hence, we have a family of closed curves spanning the set $R^2 \backslash \Omega(\alpha, k_0)$.

We shall show that these curves are contact-free almost everywhere except for the points $\{y = 0, x \in R^1\}$ and that the trajectories of system (1) "pierce" these curves from the outside to the inside. For this purpose, we shall make use of the Chaplygin–Kamke comparison principle [4–7] and inequality (2)

$$\frac{dy}{dx} = \frac{-\mu(F(y) - E(t)) - x}{y} < \frac{-\mu(\alpha y - k \operatorname{sign} y) - x}{y}$$

$$\forall x \in R^1, \quad \forall y \neq 0, \quad \forall t \in R^1$$

It follows from the comparison principle that the solutions $G_i(x)$ which correspond to the trajectories of the system

$$\frac{dy}{dt} = -\mu(\alpha y - k \operatorname{sign} y) - x, \quad \frac{dx}{dt} = y$$

and the solution $y(t), x(t)$ of system (1) possess the following property at the point $t = t_0, x_0 = x(t_0), y_0 = y(t_0) = G_i(x_0)$

$$dy/dx < dG_i/dx$$

The required contact-free property of the curves $y = G_i(x, \mu k)$ with respect to the vector field of system (1) follows from this. The assertion of the theorem also follows from the contact-free property of this family of curves almost everywhere.

We note that the inequalities

$$G_1(x, \mu k) \leqslant R_1(x - \mu k), \quad G_2(x, \mu k) \geqslant R_1(x + \mu k)$$

$$(R_1 = -\alpha\mu/2 + [(\alpha\mu)^2/4 - 1]^{1/2})$$

are satisfied.

Hence, the set $\Omega(\alpha, k)$ is located in the strip

$$\{\, |y| \leqslant 2k/(\alpha - \mu^{-1}), \ x \in R^1 \}$$

In particular, the well-known result of Cartwright [1–3] follows from this. This is concerned with the fact that the global attractor of system (1) is uniformly bounded along the y-coordinate when the parameter $\mu \in (0, +\infty)$ is varied.

We now consider the set $\Omega_0 = \cap_{\alpha,k} \Omega(\alpha, k)$, where the intersection is taken with respect to all the parameters α and k which satisfy condition (2).

Corollary 1. The global attractor of system (1) is contained in the set Ω_0.

We shall show how it is possible to apply this assertion to the simplest example when $E(t) \equiv 0$ and $F(y)$ is a Van der Pol non-linearity.

$$F(y) = y^3/3 - y \tag{6}$$

Initially putting $\alpha = 2/\mu + \varepsilon$, where ε is an arbitrarily small positive number, we obtain that the conditions (2) are satisfied when

$$k = \frac{2}{3}\left(\frac{2}{\mu} + 2\varepsilon + 1\right)^{3/2}$$

Hence, the set Ω_0 is located in the strip

$$\left\{|x| \leqslant \frac{2\mu}{3}\left(\frac{2}{\mu} + 1\right)^{3/2},\ y \in R^1\right\}$$

Next, assuming that $\mu > 2/3$, we put $\alpha = 3$. Conditions (2) are then satisfied for any $k > 16/3$. Hence, the set Ω_0 is located in the parallogram

$$\left\{|x| \leqslant \frac{16}{3}\mu,\ \left|y + \frac{x}{3\mu - 1}\right| \leqslant \frac{16}{3}\frac{1}{3 - \mu^{-1}}\right\}$$

Finally, we obtain the inclusion

$$\Omega_0 \subset \left\{|x| \leqslant \frac{2\mu}{3}\left(\frac{2}{\mu} + 1\right)^{3/2},\ \left|y + \frac{x}{3\mu - 1}\right| \leqslant \frac{16}{3}\frac{1}{3 - \mu^{-1}}\right\}$$

The latter estimate is asymptotically exact in the sense that, when $\mu \to \infty$, the boundary of this set is as close as desired to a certain part of the classical relaxation oscillation of the Van der Pol equation [1] (Fig. 2).

Now, let the inequalities

$$\beta\mu > 2,\quad \frac{F(y) - E(t)}{y} < \frac{-\beta y + \nu \operatorname{sign} y}{y},\quad \forall t \in R^1,\quad \forall y \neq 0,\quad |y| \leqslant y_0 \tag{7}$$

be satisfied (β, ν and γ_0 are certain positive numbers). We shall consider the solution $G_3(x, \mu\nu)$ of the equation

$$GdG/dx = \mu\beta G - x - \mu\nu$$

with the initial data $G_3(\mu\nu, \mu\nu) = 0$ and the solution $G_4(x, \mu\nu)$ of the equation

$$GdG/dx = \mu\beta G - x + \mu\nu$$

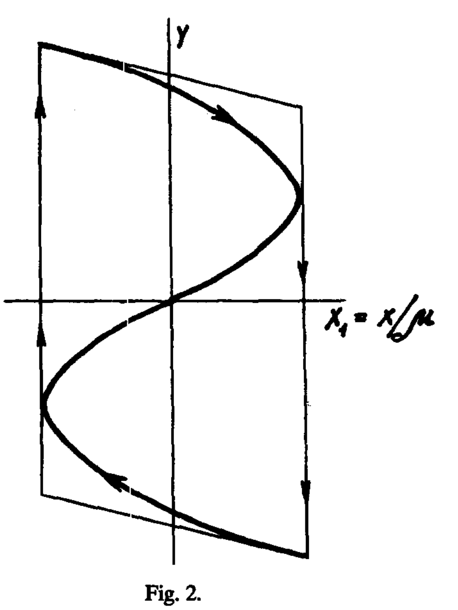

Fig. 2.

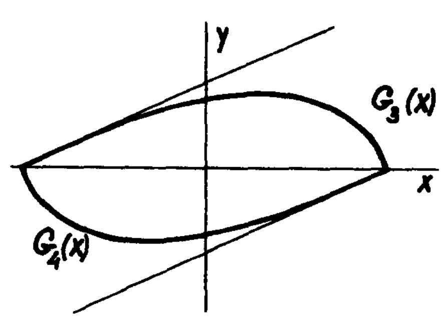

Fig. 3.

with the initial data $G_4(-\mu\nu, \mu\nu) = 0$ (Fig. 3). We note that, as in the case of Eq. (5), these equations correspond to linear second-order equations of type (3) and (4).

The set

$$\Phi(\nu) = \{x \in [-\mu\nu, \mu\nu],\ G_4(x,\mu\nu) \leqslant y \leqslant G_3(x,\mu\nu)\}$$

is now introduced into the discussion.

Theorem 2. Let inequalities (2), (7) and

$$|G_3(x,\mu\nu)| \leqslant y_0,\quad |G_4(x,\mu\nu)| \leqslant y_0,\quad \forall x[-\mu\nu,\mu\nu]$$

be satisfied.

The set $\Omega(k)\backslash\Phi(\nu)$ is then positively invariant in the case of the solutions of system (1).

The proof of this theorem repeats the arguments of Theorem 1. Here, the contact-free property of the boundary of the set Φ when $y \neq 0$ also follows from the Chaplygin–Kamke principle and from inequality (7).

Corollary 2. If a function $E(t)$ is periodic and the conditions of Theorem 2 are satisfied, system (1) does not possess the property of convergence, that is, of the stability of the periodic solution as a whole.

In fact, the set Φ is negatively invariant and, according to Brauer's theorem, a periodic solution exists in Φ.

It is clear that the existence of this solution in the set Φ and the positive invariance of the annulus $\Omega\backslash\Phi$ are at variance with the convergence property. Note that the following simple inequalities are sometimes useful for verifying the conditions of Theorem 2

$$R_2(x-\mu\nu) \leqslant G_4(x,\mu\nu) \leqslant G_3(x,\mu\nu) \leqslant R_2(x+\mu\nu),\quad \forall x \in [-\mu\nu,\mu\nu]$$

$$(R_2 = \beta\mu/2 - [(\beta\mu)^2/4 - 1]^{1/2})$$

We now consider the Van der Pol non-linearity (6) and $E(t) = b \sin \omega t$. Here, b and ω are positive numbers. Putting $\nu = b$, we obtain simple sufficient conditions for system (1) to be non-convergent

$$\beta\mu > 2,\quad 2b/(\beta - \mu^{-1}) \leqslant [3(1-\beta)]^{1/2} \tag{8}$$

For large μ, assuming that $\beta = 2/3$, we obtain the estimate $b \leqslant 1/3$.

We note that the following estimate of the boundary of the convergence domain is known [8, 3] for large μ: $b = b_0$: $b_0 \in (2/3 - 0.01, 2/3)$.

For a continuous Levinson non-linearity [3] $F(y)$

$$\frac{dF}{dy} = \begin{cases} 1, & |y| > 1 \\ -1, & |y| < 1 \end{cases}$$

and instead of the inequalities (8), we obtain the condition

$$\mu > 2,\quad b \leqslant (1 - \mu^{-1})/2$$

The necessary apparatus for extending the estimates suggested here to the case when $\mu\alpha < 2$ and $\mu\beta < 2$ has been developed in [9, 10].

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